Twenty Challenges for Computerized Symbolic Algebra Systems

A factorial identity:

0. Simplify $(x^2)((x-1)!)^2 - (x!)^2$. DERIVE gets 0 at once. MACSYMA requires that MINFACTORIAL be in force to get it. MATHEMATICA has first to be told that $x! = x\ (x-1)!$.

A Jump:

1. Simplify $A(x) := \arctan(x) + \arctan(1/x)$ to $\mathrm{sign}(\mathrm{Re}(x))\ \pi/2$. DERIVE gets $\mathrm{sign}(x)\ \pi/2$, correct only for real x . Like most systems, MATHEMATICA leaves $A(x)$ unsimplified.

2. Before simplifying $A(x)$ above, differentiate it to get a rational expression, and simplify that. Like most systems, DERIVE and MATHEMATICA simplify $dA(x)/dx$ to 0 without noticing that this is wrong when $x^2 \le 0$.

Surds:

3. Simplify $\sqrt{(x)}\sqrt{(y)} - \sqrt{(x\ y)}$ when x and y are real or complex. DERIVE leaves it alone, which is correct, unless either x or y is nonnegative, in which case DERIVE gets 0 , which is correct then. MATHEMATICA gets 0 regardless of the puzzlement caused when $x = y = -1$.

4. Simplify $\sqrt{(\sqrt{(p^4+1)} + 1)}\sqrt{(\sqrt{(p^4+1)} - 1)} - p^2$ to 0 for real p . DERIVE gets it. but MACSYMA and MATHEMATICA can't.

5. Evaluate $\int\sqrt{((x+1)/(x-1))}\ dx$, assuming real variables. DERIVE gets a correct (even for $x < 1$) result
$$(x-1)\sqrt{((x+1)/(x-1))} - 2\ \ln((\sqrt{((x+1)/(x-1))} - 1)\sqrt{(x-1)})\ ;$$
but then DERIVE cannot simplify its derivative $d(\ldots)/dx$ to recover the original integrand.

6. Simplify $\cosh(\sqrt{(-z)}) - \cos(\sqrt{z})$ to 0 for all complex z , but not $\sinh(\sqrt{(-z)}) - \imath \sin(\sqrt{z})$. ($\imath = \sqrt{-1}$) DERIVE can't do the first. The second vanishes only if $\mathrm{Arg}(z) \le 0$, but some systems "simplify" it to 0 for all z .

Two limits:

7. $$H(x) = \frac{\ln(x-a)}{(a-b)(a-c)} + \frac{\ln(x-b)}{(b-c)(b-a)} + \frac{\ln(x-c)}{(c-a)(c-b)}$$
$$= \int dx/((x-a)(x-b)(x-c))\ .$$

   Evaluate $\lim H(x)$ as $x \to \pm\infty$. The right answer is 0 . DERIVE gets it, and so does MACSYMA after the TLIMIT command. MATHEMATICA gets the expression
$$\frac{\mathrm{INFINITY}}{(a-b)(a-c)} + \frac{\mathrm{INFINITY}}{(b-c)(b-a)} + \frac{\mathrm{INFINITY}}{(c-a)(c-b)}$$
at first, and then simplifies it to 0 , which is an instance of the right answer for the wrong reason. I wonder what MATHEMATICA does with the limit of
$K(x) = 2(c-b)\ \ln(x-a) + (a-c)\ \ln(x^2-2bx+b^2) + 2(b-a)\ \ln(x-c)$.
DERIVE gets $\lim K(+\infty) = 0$ and $\lim K(-\infty) = 2(c-a)\imath\pi$ O.K.

Integrals:

Symbolic Algebra systems tend to compute $\int x^{N-1}\,dx = x^N/N$ with perhaps a warning about $N = 0$, usually without. We would all be better served by $\int x^{N-1}\,dx = (x^N - 1)/N$ with recourse to l'Hopital's Rule for $0/0$ when $N = 0$.

8. Evaluate the indefinite integral
$$W(z) := \int (z^4 - 3z^2 + 6)\,dz/(z^6 - 5z^4 + 5z^2 + 4)$$
$$= \arctan((2z^2+1)(z^2-3)z/(z^6-3z^4+2z^2+2)) + 3\arctan z\ ,$$
and then the definite integral $W(2) - W(-2) = 5\pi/2$. DERIVE and MATHEMATICA can't find it at all. It is easy to bungle. And DERIVE can't make $6\arctan(1/2) - 2\arctan(9/13) = \pi/2$.

9. Evaluate symbolically the definite integral
$$\int_0^1 dx/(\ (x+1)(x+2)(x+3) + 1/100000\ ) = 0.08494\ldots\ .$$
DERIVE gets it; my version of MATHEMATICA doesn't.

10. Evaluate the indefinite integral
$$\int (x^2 + 2x + 1)\,dx/(x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 2)$$
or its algebraic equivalent form $\int (x+1)^2\,dx/((x+1)^6 + 1)$. The right answer is $\arctan((x+1)^3)/3$. Also try the definite integral $\int_0^1 \ldots dx = (\pi/12) - \arctan(1/8)/8 = 0.22034\ldots$. DERIVE handles the second form but not the first.

11. Evaluate for real x and z the double integrals
$$\int_x^z (\ \int_{-\infty}^{\infty} y\,dt/(t^2+y^2)\ )\,dy \quad \text{and} \quad \int_{-\infty}^{\infty} (\ \int_x^z y\,dy/(t^2+y^2)\ )\,dt\ .$$
DERIVE gets $(|z| - |x|)\pi$ correctly for both. MATHEMATICA gets both wrong and different. MACSYMA asks questions.

12. Evaluate for nonnegative x the double integral
$$V(x) := \int_0^1 \int_0^{\pi} r^2 \sin(t)\,dt\,dr/\sqrt{(x^2 + r^2 - 2x\,r\cos(t))}\ .$$
This is the negative of the gravitational potential exerted by a homogeneous solid sphere at a distance x from its center. MATHEMATICA gets it wrong. DERIVE gets correctly
$$V(x) = (\ (2-x)(1+x)|1-x| - (2+x)(1-x)|1+x|\ )/(6x)$$
$$= 1 - x^2/3 \quad \text{if } 0 \le x \le 1\ ,$$
$$= 2/(3x) \quad \text{if } x \ge 1\ .$$

13. Evaluate for real x the integrals
$$\int_{-\pi}^{\pi} dt/(1 - x/\exp(\imath t)) \qquad (\ \imath^2 = -1\ ) \quad \text{and}$$
$$\int_{-\pi}^{\pi} (1 - x\cos(t))\,dt/(1 + x^2 - 2x\cos(t))\ .$$
DERIVE gets correctly $\pi - \pi\,\mathrm{sign}((x+1)/(x-1))$. MATHEMATICA gets 0 unless x has a numerical value. MACSYMA asks a blizzard of questions about x, some of them irrelevant but difficult; if answers are inconsistent it puts out an utterly wrong result 4π.

Improper Integrals:

14. Many systems compute $\int_{-1}^{1} dx/x^2 = -2$ with no warning. And a system that tries to detect improper integrals can fail. For example let $E(x) := e^x - x^e$, so $E'(x) = e^x - e x^{e-1}$ and $E''(x) = e^x - e(e-1)x^{e-2}$. Try to compute $\int_0^3 E'(x)\,dx/E(x)^{N+1}$ and $\int_0^3 E''(x)\,dx/E'(x)^{N+1}$ to see whether their impropriety will be detected when $N > 0$.

Simple inequalities:

15. Suppose $x_1 > 0$ and that $x_{n+1} = |x_n| - x_{n-1}$ for $n > 0$. (Cf. M. Brown (1985) Amer. Math. Monthly v. 92, p. 218.) Deduce that $x_9 = x_0$ and $x_{10} = x_1$. The proof can be broken into cases according as x_0/x_1 lies in one of the intervals into which the real axis is broken by the values -2, -1, $-1/2$, 0, $1/2$, 1, 2. How few such breaks does your system need? DERIVE gets by with breaks at 0 and 1.

Inverses of even complex functions:

16. Simplify $(\text{arccosh}(z))^2 + (\text{arccos}(z))^2$ to 0. Many systems can't do it. Old versions of MACSYMA have a faulty definition for arccosh ; newer versions use ATRIGHSWITCH.

A derivative:

17. Simplify $(d/dx)^n \cos(n \arccos(x))$; it should be $2^{n-1} n!$. DERIVE gets it for small integers n if $-1 \le x \le 1$. The general case requires either an unobvious induction or recognition of Tchebysheff polynomials.

Graph plotting:

18. This is the second line of defence against improper integrals, so graphs with bumps should excite curiosity. Many systems limit themselves to the harware floating-point when plotting. This can mislead spectators when $S(x) := |B + x| - B$ is plotted over, say, $0 < x < 7$ for extremely big values B, say $B = 2^{53}$ or 2^{56}. Is $S(x)$ really a step function?

19. Expressions so simple as $y(x) := 1 + x^2 + \ln(|1 - 3(x-1)|)/80$ plotted over $0 < x < 2$, roughly, have bumps that can hide from view when the plotter scatters points too sparingly to be sure of placing one near the bump. Vary the end-points of the plotting interval a little to see a spike come and go.