Elementary Functions from Kernels

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Given binary floating-point subprograms to calculate the "Kernels"

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ln(x)   for  x ≥ 0   and   ln1p(x) := ln(1+x)   for   x ≥ -1 ,
exp(x)   and   expm1(x) := exp(x)-1   for all   x ,    and
tan(x)   for   |x| < π/8   and   arctan(x)   for   |x| ≤ √2 - 1 ,
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to nearly full working accuracy, we may calculate all the other elementary transcendental functions almost as accurately, and with no violation of (weak) monotonicity, as follows. Rounding must conform to IEEE 754 or p854. We will need a threshold t chosen about as large as possible subject to the constraint that $1 - t^2$ round to 1 to working precision; and and we shall use $z := |x|$ and $s := \mathrm{copysign}(1,x) = \pm 1$. We also abbreviate expm1 to E and ln1p to L .

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sinh(x) := x  if  z < t ,  else  (provided E(z) doesn't overflow)
        := s*( E(z) + E(z)/(1+E(z)) )/2  ... certainly monotonic.

cosh(x) := 0.5*exp(z) + 0.25/(0.5*exp(z))  ...  "         "     .

tanh(x) := x  if  z < t ,  else
        := -s*E(-2*z)/(2 + E(-2*z)) .

asinh(x) := x  if  z < t ,  else,  unless  2z  overflows,
         := s*L( z + z/(1/z + √(1+(1/z)²)) ) ignoring underflow.
For slightly better accuracy when  z > 4/3 ,  use
asinh(x) := s*ln( 2z + 1/(z + √(1+z²)) )  if  z < 1/t ,  else
         := s*( ln(z) + ln(2) ) .

acosh(x) := +L( √(x-1)*(√(x-1) + √(x+1)) )  unless  2x  overflows.
For slightly better accuracy,
acosh(x) := ln(x) + ln(2)  if  x > 1/t ,  else
         := ln( 2x - 1/(x + √(x²-1)) )  if  5/4 < x ≤ 1/t ,  else
         := L( (x-1) + √(2(x-1) + (x-1)²) ) .

atanh(x) := x  if  z < t ,  else
         := s*L(2*z/(1-z))/2  .

arctan(x) := s*π/2 - arctan(1/x)  if  z > 1 ,  or (monotonically)
          := s*π/4 + arctan( (x-s)/(x+s) )  if  √2-1 < z < √2+1 .

arcsin(x) := x  if  z < t ,  else
          := arctan(x/√(1 - z²))  if  t ≤ z ≤ 1/2 ,  else
          := arctan(x/√(2(1-z)-(1-z)²)) ignoring divide-by-zero.

arccos(x) := 2*arctan(√((1-x)/(1+x))) ignoring divide-by-zero.
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For $z \le \pi/4$ let $T(x) := 2\tan(x/2)$; then
$T(x) := \tan(x) := \sin(x) := x$ and $\cos(x) := 1$ if $z < t$.
Otherwise compute tan(x), sin(x) and cos(x) thus for $z \le \pi/2$:

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tan(x) := if  z < π/8  then  T(2*x)/2
          else if  3π/8 < z  then  2s/T(π-2*z)
          else  s*(2 + T(2*z-π/2))/(2 - T(2*z-π/2)) .
     (Check monotonicity as  z  passes through  π/8  and  3π/8 .)
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If $\pi/4 \le z \le \pi/2$ then the formulas $\sin(x) = s*\cos(\pi/2-z)$ and $\cos(x) = \sin(\pi/2-z)$ reduce the argument x to y satisfying $|y| \le \pi/4$, wherein we compute $T := T(y)$, $q := T^2$, and then

$\sin(y) := y - y/(1+4/q)$;

$\cos(y) :=$ if $q < 4/15$ then $1 - 2/(1+4/q)$

else $3/4 + ((1-2*q) + q/4)/(4+q)$.

Monotonicity is preserved except possibly as x passes through multiples of $\pi/4$, where the accuracy of $T(x)$ matters.

Some implementations of $\tan(x/2)$ actually deliver two functions $A(x)$ and $B(x)$ satisfying $A(x)/B(x) = \tan(x/2)$ for $|x| \le \pi/4$, on which range $|A(x)/B(x)| < \sqrt{2} - 1 = 0.414\ldots$. These can be used to deliver sin, cos and tan more economically than above, and monotonically too provided $A(x)/B(x)$ is monotonic. For $t < z \le \pi/4$ let $r := B(x)/A(x) > \sqrt{2} + 1$; and then

$\sin(x) := 2/(r+1/r)$ and $\cos(x) := 1 - 2/(1+r^2)$.

If both of $\sin(x)$ and $\cos(x)$ are wanted simultaneously, a more economical pair of formulas is

$\sin(x) := 2/(r+1/r)$ and $\cos(x) := 1 - (1/r)\sin(x)$.

To ensure monotonicity as x passes through multiples of $\pi/4$, check that computed $\sin(\pi/4) \le$ computed $\cos(\pi/4)$; else use a better formula for cos (see above). Computing $\tan(x)$ for $|x| \le \pi/2$ from $A(x)$ and $B(x)$ is much like before:

$\tan(x) :=$ if $z < \pi/8$ then $A(2*x)/B(2x)$

else if $3\pi/8 < z$ then $B(s*\pi-2*x)/A(s*\pi-2*x)$

else $s*(B(y)+A(y))/(B(y)-A(y))$ where $y := 2*z-\pi/2$.

Monotonicity must be checked as z passes through $\pi/8$ and $3\pi/8$.

Other topics to be added later:

y^x

$\mathrm{atan2}(y,x) = \mathrm{Arg}(x + \imath y)$, especially with ± 0 and $\pm\infty$

$\mathrm{cabs}(x + \imath y) = \sqrt{x^2 + y^2}$

other complex elementary functions

approximating $\tan(z)$ for $0 < z < \pi/8$

$\arctan(z)$ for $0 < z \le \sqrt{2} - 1$

ln1p(x) and ln(x) and expm1(x) and exp(x)

argument reduction

Given $A(x)$ and $B(x)$ above, which is better:

$r := B(x)/A(x)$ and then compute $1/r$, or

$r := B(x)/A(x)$; $(1/r) := A(x)/B(x)$; ?

What is wrong with

$v := 2A/(A^2+B^2)$; $\sin(x) := vB$; $\cos(x) := 1 - vA$; ?