Elementary Functions from Kernels

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Given binary floating-point subprograms to calculate the "Kernels"

> $\ln(x)$ for $x \ge 0$ and $\mathrm{ln1p}(x) := \ln(1+x)$ for $x \ge -1$,
> $\exp(x)$ and $\mathrm{expm1}(x) := \exp(x)-1$ for all x, and
> $\tan(x)$ for $|x| < \pi/8$ and $\arctan(x)$ for $|x| \le \sqrt{2}-1$,

to nearly full working accuracy, we may calculate all the other elementary transcendental functions almost as accurately, and with no violation of (weak) monotonicity, as follows. Rounding must conform to IEEE 754 or 854. We will need a threshold t chosen about as large as possible subject to the constraint that $1-t^2$ round to 1 to working precision; and and we shall use $z := |x|$ and $s := \mathrm{copysign}(1, x) = \pm 1$. We also abbreviate expm1 to E and ln1p to L.

$\sinh(x) := x$ if $z < t$, else (provided $E(z)$ doesn't overflow)
$\quad := s \times (E(z) + E(z)/(1+E(z)))/2$ ... certainly monotonic.

$\cosh(x) := 0.5 \times \exp(z) + 0.25/(0.5 \times \exp(z))$ ... certainly monotomic.

$\tanh(x) := x$ if $z < t$, else
$\quad := -s \times E(-2 \times z)/(2 + E(-2 \times z))$.

$\mathrm{asinh}(x) := x$ if $z < t$, else, unless $2z$ overflows,
$\quad := s \times L(z + z/(1/z + \sqrt{(1 + (1/z)^2)}))$ ignoring underflow.

For slightly better accuracy when $z > 4/3$, use

$\mathrm{asinh}(x) := s \times \ln(2z + 1/(z + \sqrt{(1+z^2)}))$ if $z < 1/t$, else
$\quad := s \times (\ln(z) + \ln(2))$.

$\mathrm{acosh}(x) := +L(\sqrt{(x-1)} \times (\sqrt{(x-1)} + \sqrt{(x+1)}))$ unless $2x$ overflows.

For slightly better accuracy,

$\mathrm{acosh}(x) := \ln(x) + \ln(2)$ if $x > 1/t$, else
$\quad := \ln(2x - 1/(x + \sqrt{(x^2-1)}))$ if $5/4 < x \le 1/t$, else
$\quad := L((x-1) + \sqrt{(2(x-1) + (x-1)^2)})$.

$\mathrm{atanh}(x) := x$ if $z < t$, else
$\quad := s \times L(2 \times z/(1-z))/2$.

$\arctan(x) := s \times \pi/2 - \mathit{arctan}(1/x)$ if $z > 1$, or (*monotonically*)
$\quad := s \times \pi/4 + \arctan((x-s)/(x+s))$ if $\sqrt{2}-1 < z < \sqrt{2}+1$.

$\arcsin(x) := x$ if $z < t$, else
$\quad := \arctan(x/\sqrt{(1-z^2)})$ if $t \le z \le 1/2$, else
$\quad := \arctan(x/\sqrt{(2(1-z) - (1-z)^2)})$ ignoring divide-by-zero.

$\arccos(x) := 2 \times \arctan(\sqrt{((1-x)/(1+x))})$ ignoring divide-by-zero.

For $z \le \pi/4$ let $T(x) := 2\tan(x/2)$; then

$$T(x) := \tan(x) := \sin(x) := x \text{ and } \cos(x) := 1 \text{ if } z < t.$$

Otherwise compute $\tan(x)$, $\sin(x)$ and $\cos(x)$ thus for $z \le \pi/2$:

$$\begin{aligned}\tan(x) := {} & \text{if } z < \pi/8 \text{ then } T(2\times x)/2 \\ & \text{else if } 3\pi/8 < z \text{ then } 2s/T(\pi - 2\times z) \\ & \text{else } s \times (2 + T(2\times z - \pi/2))/(2 - T(2 \times z - \pi/2)).\end{aligned}$$

(Check monotonicity as z passes through $\pi/8$ and $3\pi/8$.)

If $\pi/4 \le z \le \pi/2$ then the formulas $\sin(x) = s \times \cos(\pi/2 - z)$ and $\cos(x) = \sin(\pi/2 - z)$ reduce the argument x to y satisfying $|y| \le \pi/4$, wherein we compute $T := T(y)$, $q := T^2$, and then

$$\begin{aligned}\sin(y) &:= y - y/(1 + 4/q); \\ \cos(y) &:= \text{if } q < 4/15 \text{ then } 1 - 2/(1 + 4/q) \\ &\quad\ \text{else } 3/4 + ((1 - 2\times q) + q/4)/(4 + q).\end{aligned}$$

Monotonicity is preserved except possibly as x passes through multiples of $\pi/4$, where the accuracy of $T(x)$ matters.

Some implementations of $\tan(x/2)$ actually deliver two functions $A(x)$ and $B(x)$ satisfying $A(x)/B(x) = \tan(x/2)$ for $|x| \le \pi/4$, on which range $|A(x)/B(x)| < \sqrt{2} - 1 = 0.414\ldots$. These can be used to deliver sin, cos and tan more economically than above, and monotonically too provided A(x)/B(x) is monotonic. For

$$t < z \le \pi/4 \text{ let } r := B(x)/A(x) > \sqrt{2} + 1; \text{ and then}$$
$$\sin(x) := 2/(r + 1/r) \text{ and } \cos(x) := 1 - 2/(1 + r^2).$$

If both of $\sin(x)$ and $\cos(x)$ are wanted simultaneously, a more economical pair of formulas is

$$\sin(x) := 2/(r + 1/r) \text{ and } \cos(x) := 1 - (1/r)\sin(x).$$

To ensure monotonicity as x passes through multiples of $\pi/4$, check that computed $\sin(\pi/4) \le$ computed $\cos(\pi/4)$; else use a better formula for cos (see above). Computing $\tan(x)$ for $|x| \le \pi/2$ from $A(x)$ and $B(x)$ is much like before:

$$\begin{aligned}\tan(x) := {} & \text{if } z < \pi/8 \text{ then } A(2\times x)/B(2\times x) \\ & \text{else if } 3\pi/8 < z \text{ then } B(s\times\pi - 2\times x)/A(s\times\pi - 2\times x) \\ & \text{else } s \times (B(y) + A(y))/(B(y) - A(y)) \text{ where } y := 2\times z - \pi/2.\end{aligned}$$

Monotonicity must be checked as z passes through $\pi/8$ and $3\pi/8$.

Other topics to be added later:
y^x
$\mathrm{atan2}(y, x) = \mathrm{Arg}(x + \iota y)$, especially with ± 0 and $\pm\infty$
$\mathrm{cabs}(x + \iota y) = \sqrt{(x^2 + y^2)}$
other complex elementary functions

approximating $\tan(z)$ for $0 < z < \pi/8$
$\arctan(z)$ for $0 < z \le \sqrt{2} - 1$
$\mathrm{ln1p}(x)$ and $\ln(x)$ and $\mathrm{expm1}(x)$ and $\exp(x)$

argument reduction

Given $A(x)$ and $B(x)$ above, which is better:
$r := B(x)/A(x)$ and then compute $1/r$, or
$r := B(x)/A(x); \quad (1/r) := A(x)/B(x);$?

What is wrong with

$$v := 2A/(A^2 + B^2); \qquad \sin(x) := vB; \qquad \cos(x) := 1 - vA; ?$$