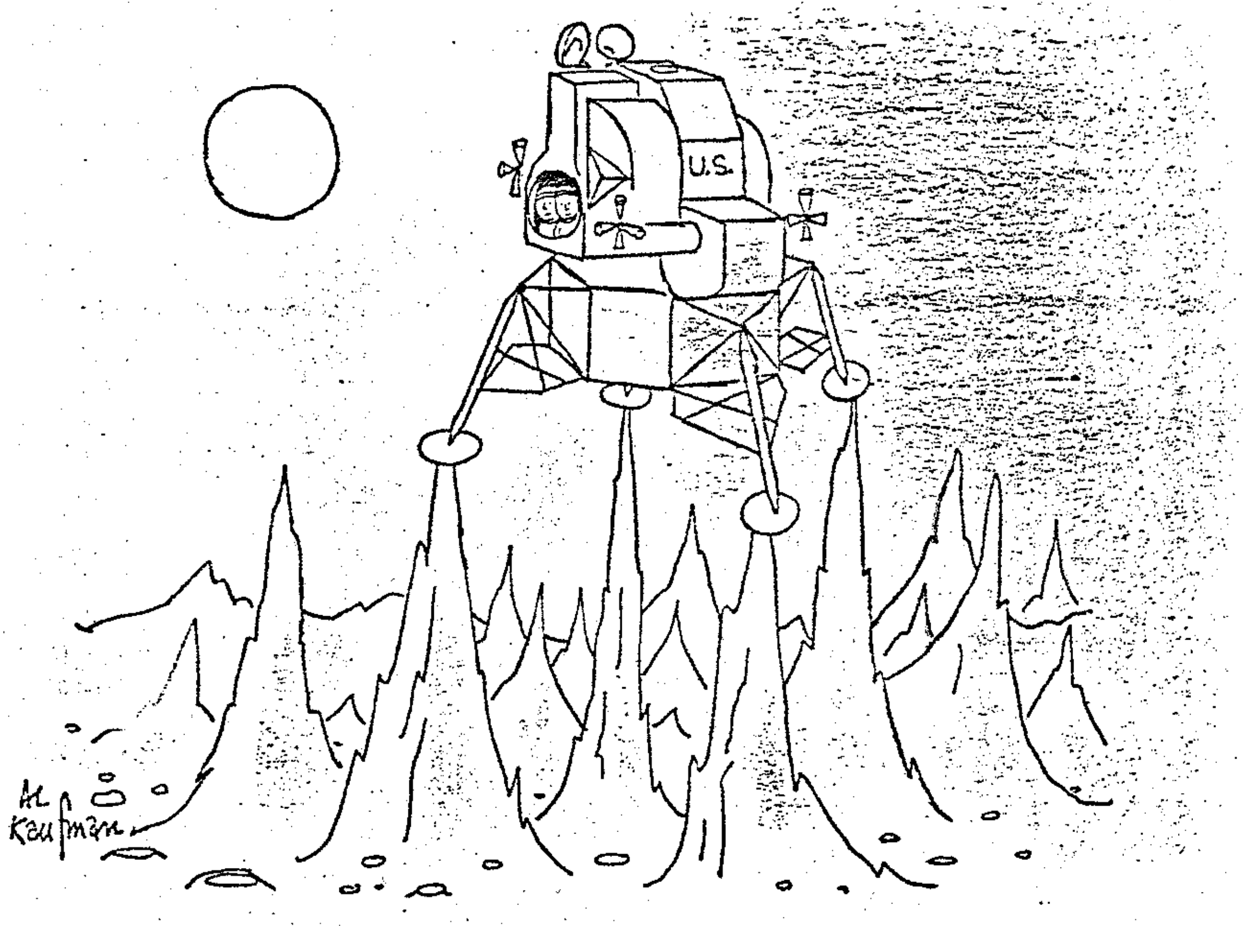
DO YOU TRUST YOUR CALCULATOR?

$$2^3 = 8.000\,000\,002$$

W. KAHAN

UNIV. OF CALIF.
BERKELEY



Kaufman

Courtesy Saturday Review

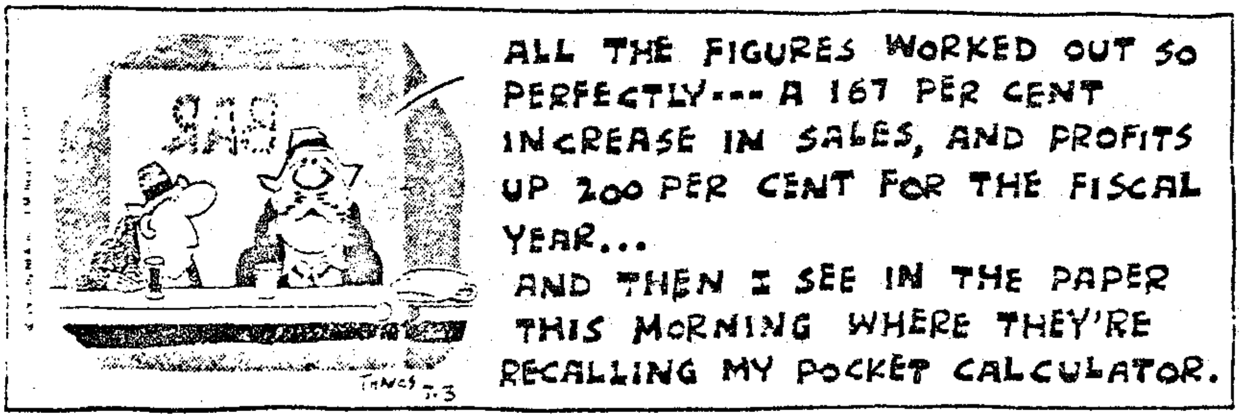
"YOU'VE GOT TO HAND IT TO THOSE COMPUTERS"

"I'VE BEEN REPLACED BY A POCKET CALCULATOR"

The INDEPENDENT and GAZETTE, Sat., July 3, 1976—7

FRANK AND ERNEST

By Thaves



HOW MANY SIG. FIG'S ?

$$X = (5 - 10^{1-n}) + 5$$

$$= \underset{\longleftarrow\; n \;\longrightarrow}{9.999\ldots999} \quad ; \qquad \text{so}$$

$$(X-5)-5 = -10^{1-n} .$$

DOES $X - 10 = -10^{1-n}$?

OR $\ldots = -10^{2-n}$? *

H-P : $= 10$ Rounded $\varepsilon \div 5\times10^{-10}$

MONROE : $= 13$ Rounded $\times \div$ $\varepsilon = 5\times10^{-13}$
Chopped $+ -$ $\varepsilon = 10\times10^{-13}$ *

TI : $= 12$ or 13 $\varepsilon \approx 10^{-12}$ **

* SOMETIMES LACK GUARD DIGITS
** ALWAYS " " "

T.I. SR-52 Something Entirely Different

Book says "12 sig. fig's", and

$$\left.\begin{array}{l} X = (5 - 10^{-12}) + 5 \\ (X - 5) - 5 \quad \Rightarrow -10^{-11} \end{array}\right\}$$ 12 Unguarded Sig Fig's.

BUT

$$5+(5-(5+(5-10^{-12}))) \Rightarrow 10^{-12}$$

IMPLIES 13 SIG. FIG'S.

$\frac{\pi}{2} - \frac{\pi}{2} \Rightarrow -5_{10}{-12}$ $\quad x \neq x$

$\left(\frac{1}{3} - \frac{3}{9}\right) + \left(\frac{3}{9} - \frac{1}{3}\right) \Rightarrow -6_{10}{-13}$ $\quad (x-y)+(y-x) \neq 0$

$e\pi - \pi e = \pi e - e\pi \Rightarrow -5_{10}{-12}$ $\quad xy \neq yx$

ACTUALLY $e\pi - \pi e = 2.8_{10}{-11}$ ← AS ON SR 50, 51

No guard digits.

Storage cells & arithmetic have 13 sig. fig's but sub-expression stack (...) has 12.

SIGNIF. DEC. DIGITS

T.I. SR-	50, 50A	51, 51A	52,
DISPLAYED ARITH.	13	13	13
SUB-EXPR'N	13	13	12
STORAGE	13	12	13

HP	...	10
MONROE	...	13

1. DOES $xy - yx \approx \pm 10^{-12} xy$ MATTER?

2. DON'T <u>ALL</u> CALCULATORS HAVE TO MAKE LITTLE ERRORS ANYWAY ?

3. ANY TEST THAT CAN BE MISINTERPRETED WILL BE.

$$\underbrace{\sqrt{\cdots\sqrt{\overset{\cdots}{\sqrt{\sqrt{2}}}}}}_{\longleftarrow\ 20\ \longrightarrow} = \sqrt[2^{20}]{2}$$

$$1.000\,000\,661\,038\ldots$$

$$\Bigg(\cdots\bigg(\bigg(\bigg(\sqrt[2^{20}]{2}\bigg)^2\bigg)^2\bigg)^2\cdots\Bigg)^2_{\longleftarrow\ 20\ \longrightarrow} = 2$$

20+20 = 40 ROUNDING ERRORS

NO CANCELLATION.

NO CATASTROPHIC LOSS OF SIGNIFICANT FIGURES.

HP	1.999 897 829
MONROE	2.000002019384
TI SR-51/2	1.999996813919
	(1.999984235727)

$$2^{20} = 1{,}048{,}576.$$

$$\underbrace{\sqrt{\cdots\sqrt{\sqrt{\sqrt{2}}}}}_{20} = \sqrt[2^{20}]{2}$$

$$\Big(\cdots\Big(\Big(\Big(\sqrt[2^{20}]{2}\overbrace{\Big)^{2}\Big)^{2}\Big)^{2}\cdots\Big)^{2}}^{20} = 2$$

BUT

$$\Big(\cdots\Big(\Big(\Big((1+\varepsilon)\sqrt[2^{20}]{2}\overbrace{\Big)^{2}\Big)^{2}\Big)^{2}\cdots\Big)^{2}}^{20} = (1+\varepsilon)^{2^{20}}\cdot 2$$

$$\doteq (1+2^{20}\varepsilon)\cdot 2$$

HP $\varepsilon \approx 10^{-10}$

TI $\varepsilon \approx 10^{-12}$

MONROE $\varepsilon \approx 10^{-13}$

MEAN, AND STANDARD DEV'N

DATA: $x_1, x_2, x_3, \cdots, x_n$

MEAN: $\bar{x}_n = \sum_1^n x_j / n$

ST'D DEV'N: $\bar{s}_n = \sqrt{\sum_1^n (x_j - \bar{x}_n)^2 / (n-1)}$

$$= \sqrt{\frac{\sum_1^n x_j^2 - n\bar{x}^2}{n-1}}$$

TEST EXAMPLE: $n = 12$

$x_1 = x_2 = x_3 = \cdots = x_{12} = 55555.55555$

$\bar{x} = 55555.55555$

$\bar{s} = 0$.

cf. also BEST STRAIGHT LIN
LINEAR REGRESSIO

SIR THOMAS GRESHAM'S LAW

1519 - 1579

"BAD MONEY DRIVES OUT GOOD"

(i.e. OUT of circulation)

where are the silver coins?

FAST PROGRAMS DRIVE OUT THE SLOW

EVEN IF THE FAST ONES ARE WRONG ...

... when the st'd dev'n is tiny compared with the mean.

USE FAST PROGRAMS ONLY WITH

3 sig. fig. data on H.P

4 sig. fig data on TI, MONROE

DATA $x_1, x_2, x_3, \dots, x_n$

MEAN $\bar{x}_n \equiv \sum_1^n x_j / n$

ST'D DEV'N $\bar{\sigma}_n \equiv \sqrt{\underline{\sum_1^n (x_j - \bar{x}_n)^2}/(n-1)}$

$$= \sqrt{\left(\sum_1^n x_j^2 - n\bar{x}_n^2\right)/(n-1)}$$

RECURRENCES FOR $\bar{x}_n$ AND $Q_n \equiv \sum_1^n (x_j - \bar{x}_n)^2$:

cf. linear regression too. + a few tricks (CS246)

$$\bar{x}_{n+1} = \bar{x}_n + (x_{n+1} - \bar{x}_n)/(n+1)$$

$$Q_{n+1} = Q_n + \frac{n}{n+1}(x_{n+1} - \bar{x}_n)^2$$

cf. statistics texts & papers of the 1940's & 1950's, & COMM. A.C.M. <u>11</u> (1968) 149-150, <u>18</u> (1975) 57-8

$x_1 = 99999\ 99990$

$x_2 = 99999\ 99991$

$x_3 = 99999\ 99992$

$x_4 = 99999\ 99993$

$\bullet\bullet\bullet \qquad \bullet\bullet\bullet$

$x_9 = 99999\ 99998$

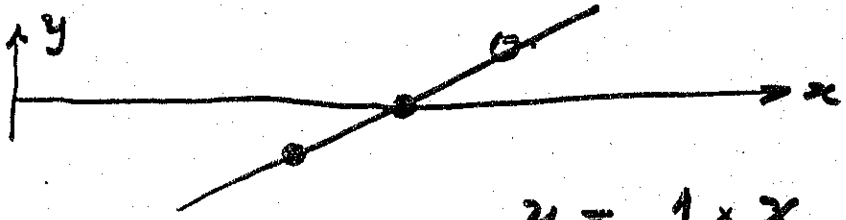
$x_{10} = 99999\ 99999$ $\qquad n=10$

$\bar{x} = 99999\ 99994.5$

ST'D DEV'N = 3.0276 50354

BEST STRAIGHT LINE FIT

THROUGH 3 COLLINEAR POINTS



$$y = 1.\times x - 666000$$

$x_1 = 665999$ $\quad y_1 = -1$

$x_2 = 666000$ $\quad y_2 = 0$

$x_3 = 666001$ $\quad y_3 = 1$

HP 91, 27, ... Error

TI SR 51 .... Blinks

HP 65 (HP prog'm) Blinks

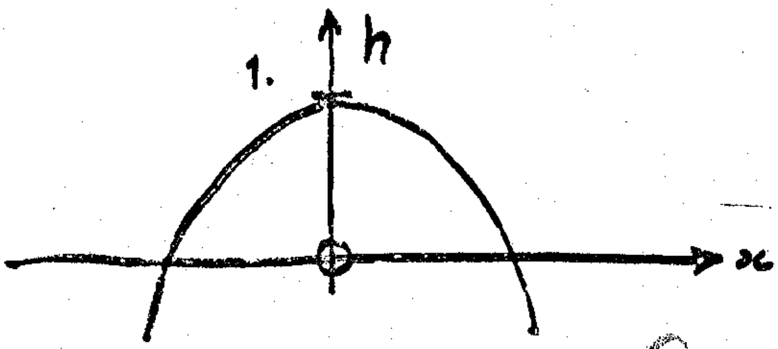
They calculate Variance(x) = 0.

HP 65, 67, 97 (My prog'm) b = −666000, m = 1 ✓

HOW TO SAVE ~~2~~ 3 REGISTER

x	y
1971	30 000
1972	32 500
1973	35 000
1974	37 500

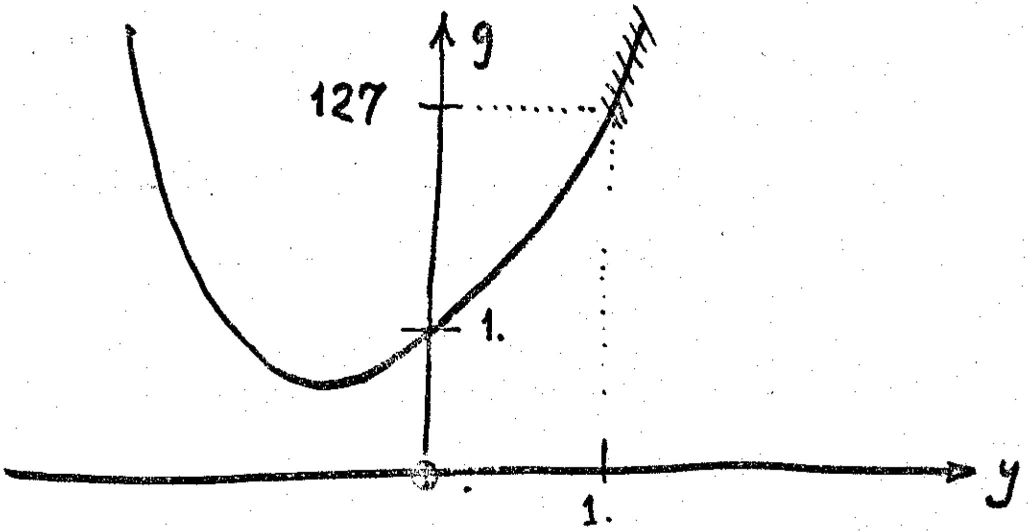
$$h(x) \equiv \left(\tfrac{1}{3} - x^2\right)\cdot\left(3 + 3.45\,x^2\right)$$

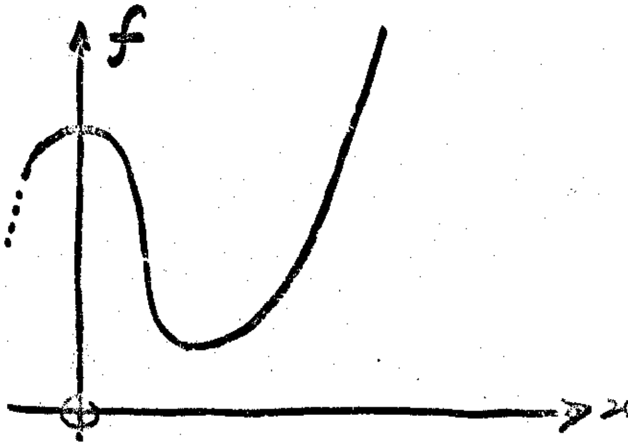


$h(x) \leq 1$

COMPUTED $h(x)$ [illegible] 1

$$g(y) \equiv 1 + y + y^2 + y^3 + \cdots + y^{125} + y^{126}$$





$$f(x) \equiv g(h(x))$$

POLYNOMIAL.

cf. Interest, Discount & Yield calculations

$$g(y) = 1 + y + y^2 + y^3 + \cdots + y^{125} + y^{12$$

for $y < 1$,

$$= \frac{y^{127} - 1}{y - 1} \quad . \quad \text{FASTER}$$

$$y^{127} = \exp(127 \ln y)$$

$$= y(y(y(y(y(y(y^2))^2)^2)^2)^2)^2$$

12 multiplications

$$= ((((((y^2)^2)^2)^2)^2)^2)^2 / y$$

7 multiplications & 1 division.

$$g(y) = \frac{y^{127}-1}{y-1}$$

WHAT DOES ROUNDOFF DO WHEN y IS CLOSE TO 1 ?

CASE 1 "NOT TOO CLOSE"

$$|y-1| \gtrsim \frac{\sqrt{\varepsilon}}{127}$$

HP $\varepsilon = 10^{-1}$

TI $\varepsilon = 10^{-13}$

$$\frac{y^{127}-1}{y-1} \Rightarrow \frac{\left(y^{127}(1\pm\varepsilon)-1\right)(1\pm\varepsilon)}{(y-1)(1\pm\varepsilon)}\cdot(1\pm\varepsilon)$$

$$= (1\pm 3\varepsilon)\, g(y)\left\{1 \pm \frac{\varepsilon}{y^{127}-1}\right\}$$

But $|y-1| \gtrsim \frac{\sqrt{\varepsilon}}{127} \Rightarrow |y^{127}-1| \gtrsim \sqrt{\varepsilon}$

$$\frac{y^{127}-1}{y-1} \Rightarrow g(y)\left\{1 \pm \sqrt{\varepsilon}\right\}$$

AT MOST ABOUT HALF THE FIGURES CARRIED ARE LOST

$$g(y) = \frac{y^{127}-1}{y-1}$$

CASE 2 "VERY CLOSE"

$$y = 1-\delta \quad \text{AND} \quad 0 < \delta \lesssim \frac{\sqrt{\varepsilon}}{127}$$

$$y^{127} = (1-\delta)^{127} = 1 - 127\delta + \underbrace{127 \times 63\,\delta^2}_{\lesssim \frac{1}{2}\varepsilon} - \cdots$$

$$\Rightarrow 1 - 127\delta$$ IF CORRECTLY ROUNDED OR CHOPPED

$$\therefore \quad \frac{y^{127}-1}{y-1} \Rightarrow \frac{(1-127\delta)-1}{(1-\delta)-1} = 127$$

$$\text{But } g(y) = 127\left(1 - \underbrace{63\delta + \cdots}_{\text{Rel. Error} \lesssim \frac{1}{2}\sqrt{\varepsilon}}\right)$$

AGAIN, AT MOST ABOUT HALF THE FIGURES CARRIED ARE LOST

$$f(x) = g(h(x))$$

$$h(x) = \left(\frac{1}{3} - x^2\right)\cdot\left(3 + 3.45x^2\right)$$

$$g(y) = \frac{y^{127} - 1}{y - 1}$$

TRY TINY VALUES OF x.

$x = 0$:	MONROE 326	12.
	TI SR-50, SR-51	14.
	SR-52	128.
	HP	127.

$y-1$ when $y = .999\ldots999$

$$\begin{array}{rcl} -1 & = & -1.00 \;\ldots\; 000 \\ y & = & 0.99 \;\ldots\; 999\cancel{9} \\ \hline & & -0.00 \;\ldots\; 001\cancel{0} \end{array}$$

∴ REPLACE $y-1$

BY $(y-0.5)-0.5$?

- - - - - - - - - - - -

$x=0$:	Monroe 326		120
	TI	SR-50	13!
		SR-51	128
		SR-52	1271
	HP		127

For tiny x

$$\epsilon^x = 1 + x + \tfrac{1}{2}x^2 + \cdots$$

$$\Rightarrow 1 + x \quad \text{when rounded.}$$

$$y^x = \epsilon^{x \ln y}$$

REPLACE y^{127} by $((((((y^2)^2)^2)^2)^2)^2)^2/y$

MONROE	326	127	
TI	SR-50, 51	1270	*
	SR-52	1279	
HP		127	

* No Guard digit in Multiplication, Division.

The POLYNOMIAL $f(x) = g(h(x))$ CAN BE CALCULATED

NAIVELY, IN FEW OPERATIONS, CORRECT TO 5 sig. fig's ON ANY HP CALCULATOR

WITH A FEW TRICKS & A FEW MORE OPERATIONS, TO 6 sig. fig's ON MONROE 324, 325, 326, & T.I. SR-52

NOT EASILY USING "13 sig. fig. Arithmetic" ON T.I. SR-50 OR SR-51.

THE MORAL: 10 CLEAN FIGURES CAN BE WORTH MORE THAN 13 DIRTY ONES.

NATURAL LOGARITHM

$$\ln(x) = (x-1) - \tfrac{1}{2}(x-1)^2 + \cdots$$

when $|x-1| < 1$

x	.9999995	.999995
HP 21, 25, 35 45, 55, 65	$-5_{10}-7$	$-5_{10}-6$
HP 22, 91, 27, 67, 97	$-5.00000125_{10}-7$	$-5.0000125_{10}-6$
MONROE 326	$-5.00000125_{10}-7$	$-5.0000125_{10}-6$
T.I. SR-50, 51, 52	$-5_{10}-7$	-0.000005 or $-5.0000125_{10}-6$ *

* Output format depend
upon input format,
.999995 or $.999995_{10}0$

$$\tan(90^\circ + x) = -\tan(90^\circ - x)$$

	tan(x)	
x	90.000001°	89.999999°
HP 21, 25, 45, 55, 65	-58823529.41	58363500.16
HP 27, 67, 91, 97	-57295779.51	57295779.51
MONROE 326	-57,295,779.51	57,295,779.51 *
T.I. SR-50, 51, 52	-57300361.53	57290501.51

* MONROE MACHINES HAVE TRIG FUNCTIONS IN DEGREES BUT NOT RADIANS.

DOES $\sin 2\theta = 2 \sin\theta \cos\theta$?

Try $\theta = 52174$ radians : *

MACHINE	$\sin 2\theta$	$2 \sin\theta \cos\theta$
TI SR 5X	$-1.099\,5573_{10}-5$	$-1.100\,81398\,2_{10}-5$
HP 25, 45, 55, 65	$2.610\,169797_{10}-6$	$2.610\,238060_{10}-6$
HP 27, 91, 97, 67	$-1.100815000_{10}-5$	$-1.100\,815000_{10}-5$
∞	$-1.1015\,01758\ldots_{10}-5$	$-1.1015\,01758\ldots_{10}-5$

---

in Degrees

DOES $\sin(2\times 10^n) = \sin(200)$ for $n = 2, 3, \ldots$

$= -.342020143\ldots$?

MONROE 326
HP 27, 67, 91, 97 } YES for all n.

HP 25, 45, 55, 65
TI SR 5X } NOT for large n

---

* MONROE 326 DOES NOT PROVIDE TRIG (RADIANS).

Is $\sin(2\theta) \doteqdot 2\sin(\theta)\cos(\theta)$?

$\theta = 3141592654$ DEGREES

	HP91,27,...	HP65etc.	SR·5x	MONROE 326
SIN(2θ) 2 SIN(θ)cos(θ)	EXACT	1 sig. fig	4 sig. fig.	EXACT

$\theta = 3141592654$ RADIANS

	HP91,27,...	HP65etc.	SR-5x	MONROE 326
SIN(2θ) 2 SIN(θ) COS(θ)	MATCH ENTIRELY	0 0	3 sig. fig's MATCH	X
	3 sig. fig's correct		3 sig fig's Correct.	

$10^9\pi = 3141592653.58979323846\ldots$

$\sin(2\theta) = .731427881\ldots$

$$X^2 - 3\times10^n X + 1 = 0$$

Roots , for larger n

about 3×10^n

$.333333\cdots \times 10^{-n}$

TEST SOFTWARE FOR QUADRATIC EQUATION.

$$Ax^2 + Bx + C = 0$$

$$X_1 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

CHOOSE SIGN

$$X_2 = \frac{C}{AX_1}$$

$$\mathrm{SIGN}(B) = B/|B|$$

Try $x^2 - 1 = 0$!

$$\frac{e^{y}-e^{z}}{y-z}=e^{\frac{y+z}{2}}\ \frac{\sinh\left(\frac{y-z}{2}\right)}{\frac{y-z}{2}}$$

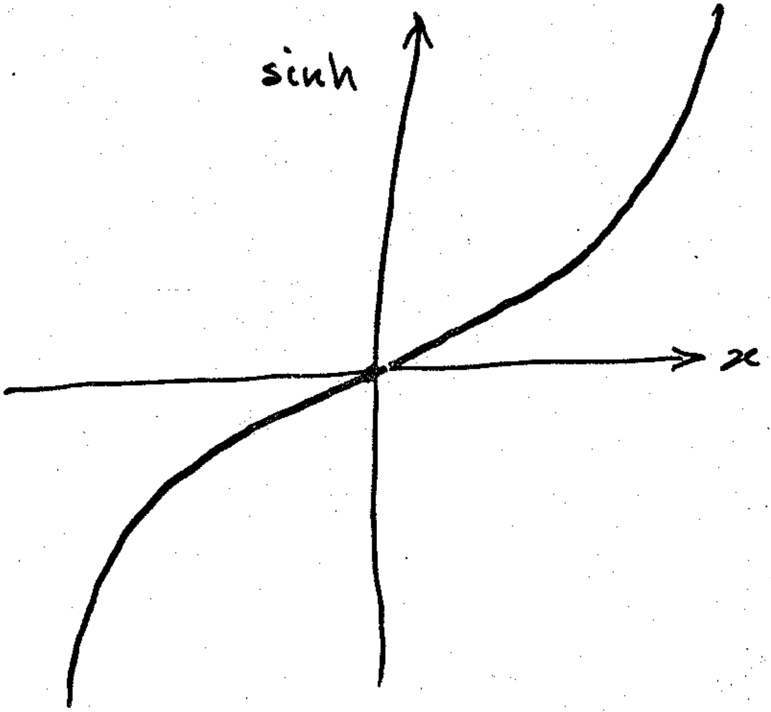
Useful for Linear
Ordin'y Diff'l Equ'ns.

$$\sinh(x)=\frac{e^{x}-e^{-x}}{2}$$

CANCEL
WHEN
x
IS TINY!

$$=\sum_{0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

$$\frac{\sinh(x)}{x}\geqslant 1 \quad \text{for all } x$$



$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

WHEN $x^2 < 0.3$,

$$\sinh(x) = x + x^3 \Big/ \left(6 - 3x^2 \Big/ \left(10 + x^2 \Big/ \left(7 + \frac{294}{x^2 - 92.4}\right)\right)\right)$$

with REL. ERROR $< 10^{-9}$

"MACHINE-INDEPENDENT" UNIFORM FORMULA :

$$y = e^{-|x|} \quad \text{ROUNDED}$$

If $y = 1$ set $\sinh(x) = x$,

else $$\sinh(x) = \frac{((y - 0.5) - 0.5) \cdot \left(0.5 + \frac{0.5}{y}\right)}{\ln y}$$

with same REL. UNCERTAINTY as $\ln$

5 sig. fig's on TI SR-... & HP... ≠ 91, 22 or 27

Full display accuracy on HP $\underset{67,\,97}{22,\,91}$, 27, MONROE 326.

$$\frac{\sinh(x)}{x} \underset{?}{=} 1 + \frac{x^2}{6} + \frac{x^4}{120} + \cdots$$

x	$\sqrt{3 \times 10^{-9}} =$ $5.477225575_{10}-5$	$_{10}-5$	$1.23456_{10}-11$
HP 65 HP's way	1.000 004459	.999 995	O
HP 65 use ln	1.	1.000 005 000	1.
HP 65 CONT. FR'N	1.000 000 001	1.	1.
SR 50	.999 999 995 4	1.000 000 050	1.053
SR 51	.999 999 995 4	1.000 000 050	1.
SR 52 TI's way	.999 999 968 0	1	.972
MONROE 326 MONROE'S WAY	.999 999 986 2	1	.972
MONROE 326 use ln	1.000 000 000 501	1.000000 000 017	.99999999999 43
HP 27,67,91,97 use ln	1.000 000 001	1.	1.

MONROE 326 LINEAR EQUATIONS SOLVER

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 10^{-15} & 1 & 0 & -10^{-15} \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{array} \cdot \begin{array}{c} \\ \begin{pmatrix} 2 \\ 1 \\ 0 \\ -2 \end{pmatrix} \end{array} = \begin{array}{c} 5 \\ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{array}$$

$$\underset{\text{INPUT}}{A} \cdot \underset{\text{OUTPUT}}{x} = \underset{\text{INPUT}}{b}$$

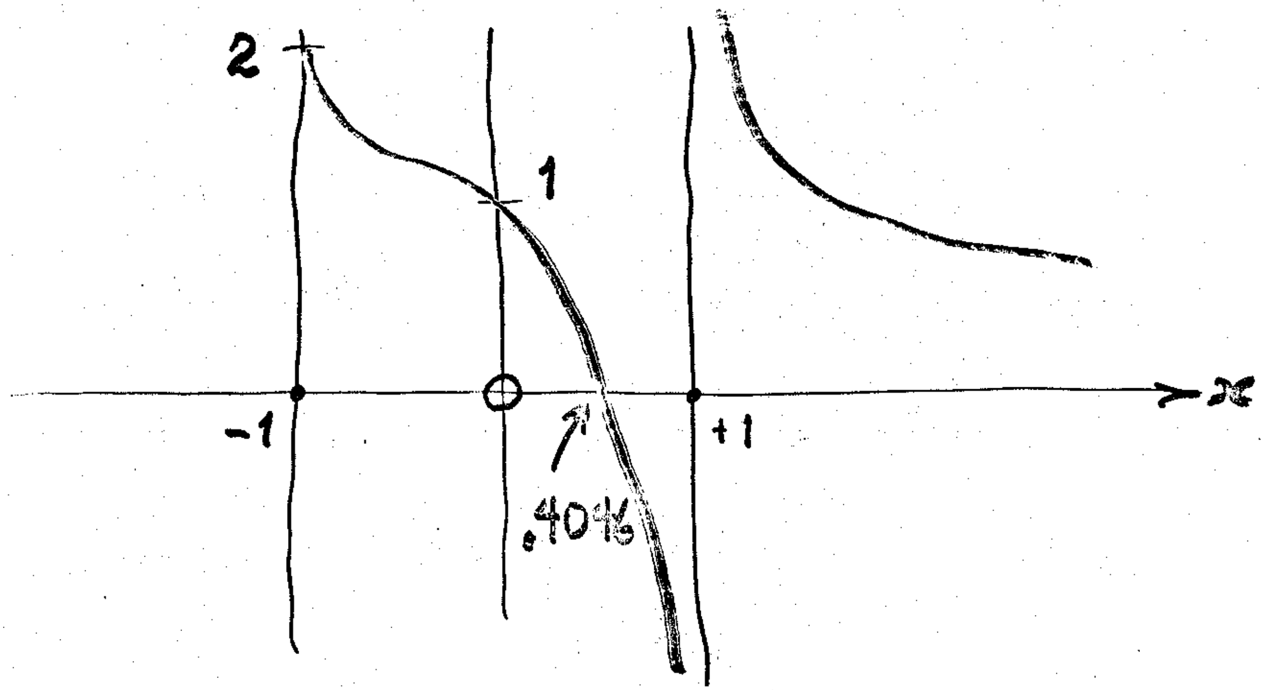
$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} 10^{-15} & 1 & 0 & -10^{-15} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

SOLVE $E(x) = 0$ for x.

$$E(x) = 1 \text{ if } x = 0,$$

$$\text{else} = 2 - \frac{x}{(1-x)\ln(1+x)}$$

$$\text{for } -1 \leq x < 1$$



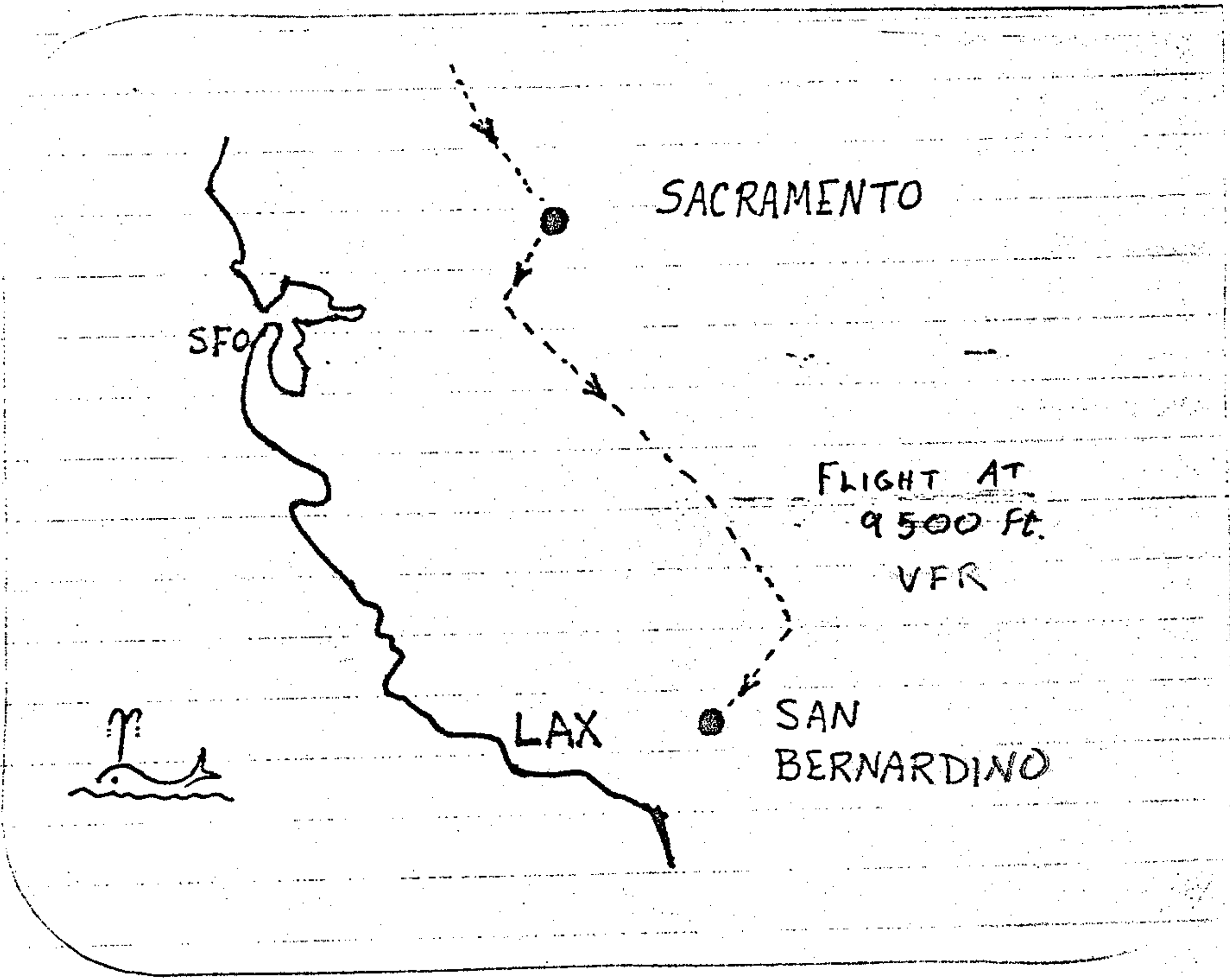
MONROE 326 "Newton - Raphson" Iteration (Secant

CONVERGES O.K. from $x = 1 - \frac{1}{120} - 10^{-4}$

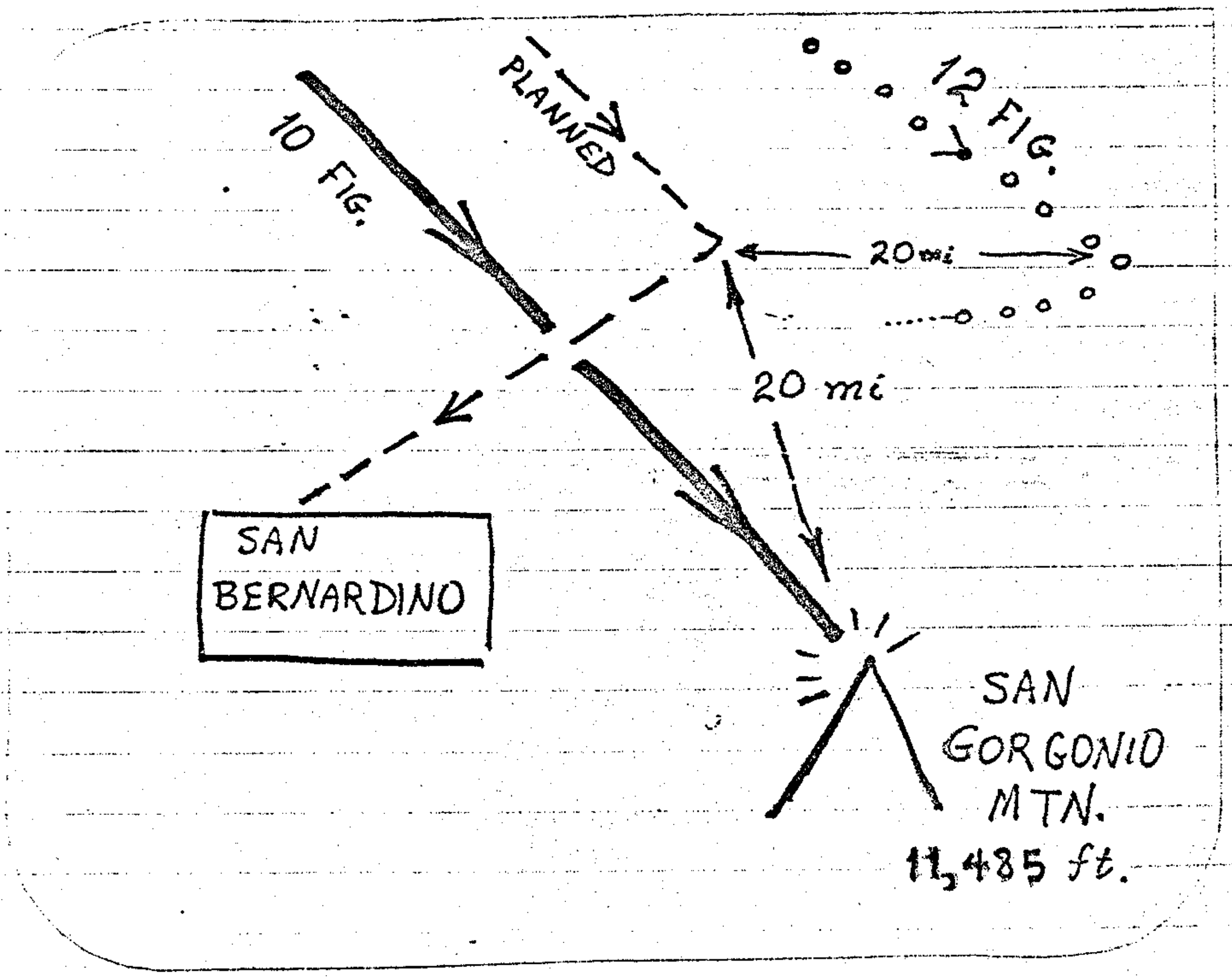
HANGS UP FAST ... $1 - \frac{1}{120}$

HANGS UP AFTER 30 min ... $1 - \frac{1}{120} + 10^{-13}$

HP-65 STRADDLING SECANT METHOD CAN "CONVERGE" to UTTERLY WRONG ANSWER, O JUMP to $x \leq -1$ from inside $-1 < x < 1$.


SACRAMENTO
SFO
FLIGHT AT
9500 Ft.
VFR
LAX
SAN
BERNARDINO

FLIGHT @ 9,500 ft , VFR



LEGAL PRECEDENTS

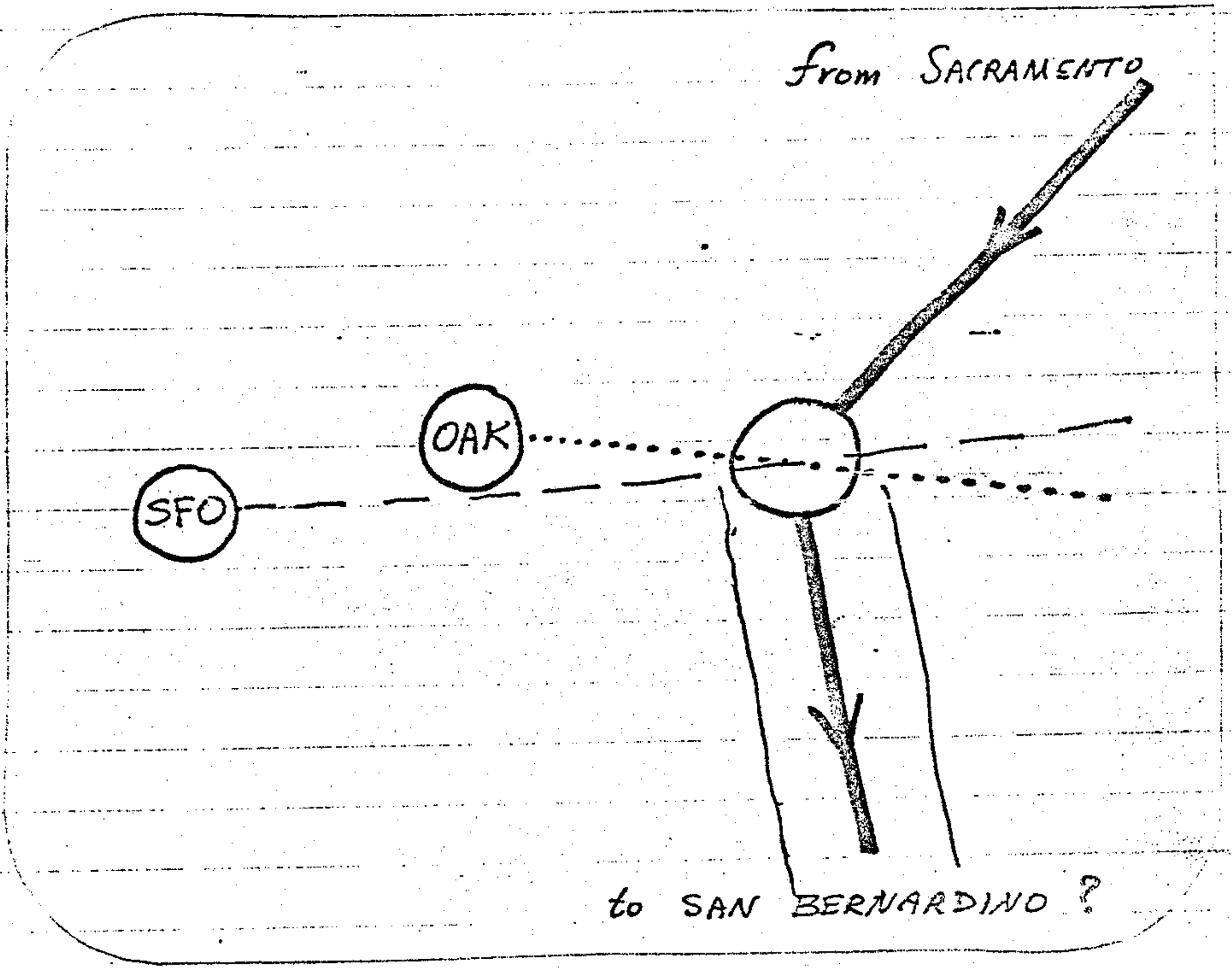
CONCERNING PRIVITY
WARRANTY
LIABILITY

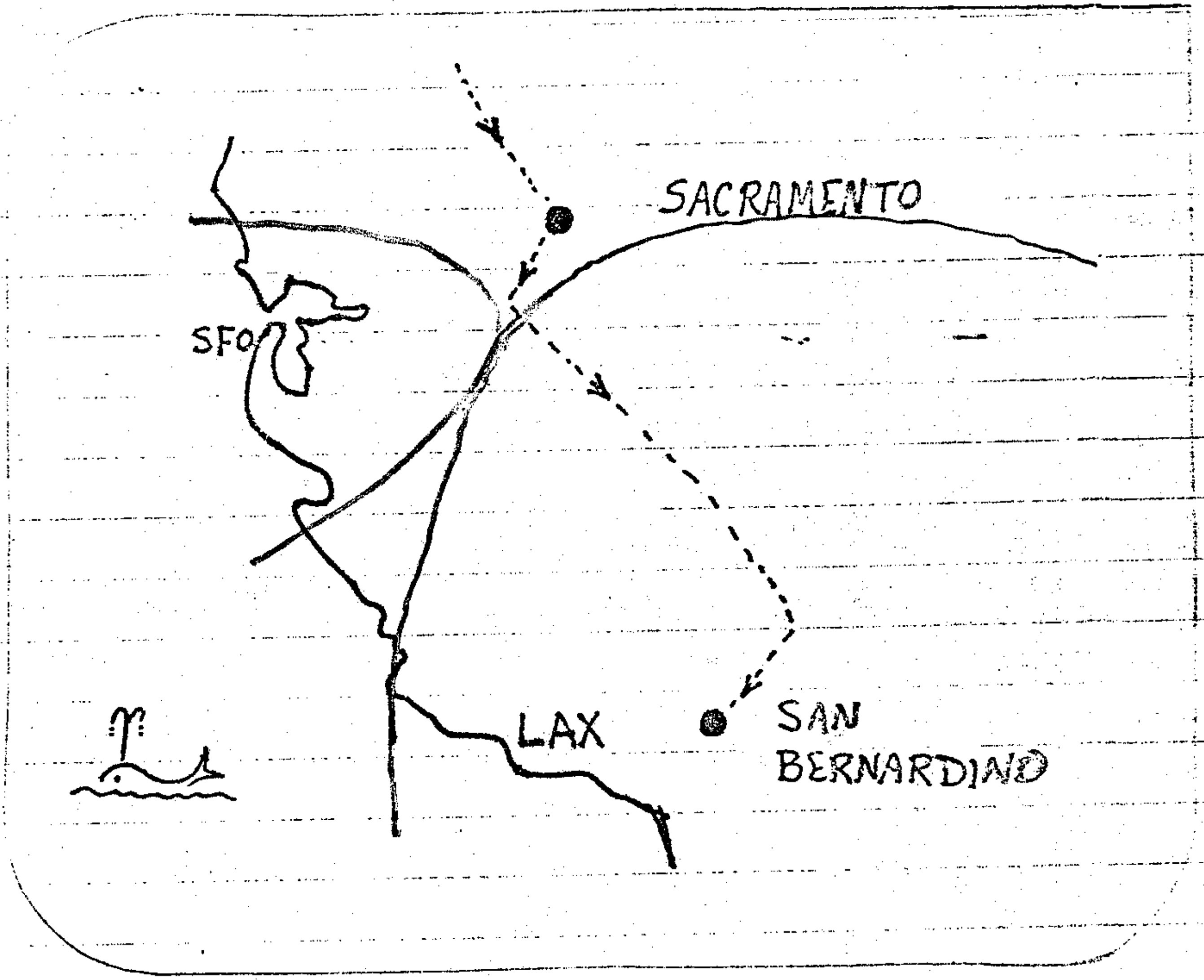
1960: GOTTSDANKER vs. CUTTER LABS
PHIPPS vs. CUTTER LABS

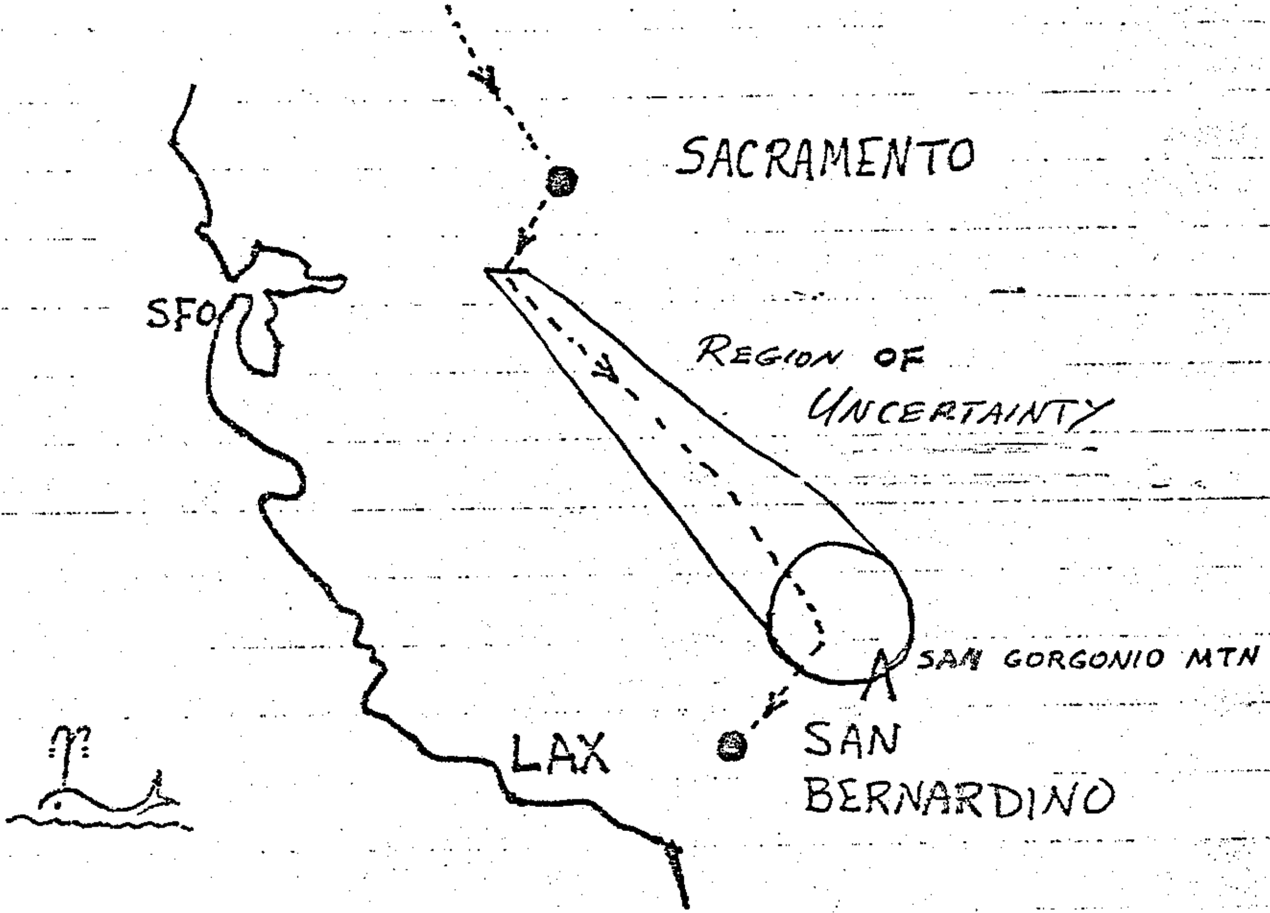
Anti-polio vaccine proves defective & causes polio. Defendant found NOT GUILTY of NEGLIGENCE, but LIABLE under IMPLIED WARRANTY

1968: DAVIS vs. WYETH LABS

Anti-polio vaccine induces small ($< 10^{-6}$) risk of polio in adults. Plaintiff was not warned of this "Negligible" risk; Defendant found LIABLE under IMPLIED WARRANTY in lieu of adequate warning.


from SACRAMENTO
OAK
SFO
to SAN BERNARDINO ?


SACRAMENTO
SFO
LAX
SAN
BERNARDINO


SACRAMENTO
SFO
REGION OF UNCERTAINTY
SAN GORGONIO MTN
LAX
SAN BERNARDINO

[illegible]

"NOBODY EVER LOST MONEY
BY UNDERESTIMATING THE
INTELLIGENCE OF THE
AMERICAN CONSUMER."

vs.

AN INFORMED CONSUMER
IN A FREE MARKET.

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