WHERE DOES LAGUERRE'S METHOD COME FROM?

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Given only the numerical values of $N, x, P(x), P'(x)$ and $P''(x)$, where P is an N^{th} degree polynomial, where might one best begin to look for a zero of P? Laguerre's formula offers two suggestions:

$$x - \frac{P(x)/P'(x)}{1/N \pm (1-1/N)\left(1-\dfrac{NP(x)P''(x)}{(N-1)(P'(x))^2}\right)^{1/2}}$$

This formula was published by Laguerre in 1880 with a derivation, valid only for polynomials with all zeros real, which showed that then all of P's zeros are separated from x by the two suggestions, but he did not explain why either of those suggestions might be expected to be close to any of P's zeros. Moreover, that derivation was considered sufficiently obscure by Hermite, one of the editors of Laguerre's "Oeuvres" (1898), that he inserted therein another derivation of his own to clarify, as he thought, Laguerre's. In 1948 Bodewig showed that Laguerre's suggestions, when interpreted as an iteration, would converge cubically to simple zeros, again assuming all P's zeros to be real. The first derivation of Laguerre's formula for complex zeros was published by Maehly in 1954, but his derivation assumed the validity of an approximation which seems, on retrospect, to have been arbitrarily contrived so as to yield Laguerre's formula, and offers no stronger reason to use the iteration than that it converges to simple zeros cubically if it converges at all.

We offer here another derivation, based upon the maximum likelihood principle. We assume, besides the numerical values, the existence of an unspecified a-priori probability distribution for the zeros of P with no more than the following properties:

i) The probability density is continuous over the Riemann sphere of complex numbers, except possibly at x where we know $P(x) \neq 0$.

ii) The probability density is positive everywhere except possibly at x.

From these hypotheses we construct a conditional probability density for the zeros of those polynomials of degree N which match the given values at x (we cannot discriminate amongst those polynomials using only the given information). Then we compute a marginal density for any one of the zeros of that family of polynomials; and we find, after computations too complicated to reproduce here, that the marginal density is finite everywhere except at the two points given by Laguerre's formula where the density is infinite. By the maximum likelihood principle, these are the best places to look for one of P's zeros, no matter what the a-priori distribution may have been provided it satisfied i and ii above.